Part VIII

Appendices

Appendix A

Greek Letters

The following table shows the greek letters, (some of them have two typeset variants), and their corresponding Roman letters.

Name	*Roman*	*Lower*	*Upper*
alpha	a	α	
beta	b	β	
chi	c	χ	
delta	d	δ	Δ
epsilon	e	ϵ	
epsilon (variant)	e	ε	
phi	f	ϕ	Φ
phi (variant)	f	φ	
gamma	g	γ	Γ
eta	h	η	
iota	i	ι	
kappa	k	κ	
lambda	l	λ	Λ
mu	m	μ	

nu	n	ν	
omicron	o	o	
pi	p	π	Π
pi (variant)	p	ϖ	
theta	q	θ	Θ
theta (variant)	q	ϑ	
rho	r	ρ	
rho (variant)	r	ϱ	
sigma	s	σ	Σ
sigma (variant)	s	ς	
tau	t	τ	
upsilon	u	υ	Υ
omega	w	ω	Ω
xi	x	ξ	Ξ
psi	y	ψ	Ψ
zeta	z	ζ	

Appendix B

Notation

C	class of continuous functions
C^n	class of n-times continuously differentiable functions
$\mathbb{C}$	set of complex numbers
$\delta(x)$	Dirac delta function
$\mathcal{F}[\cdot]$	Fourier transform
$\mathcal{F}_c[\cdot]$	Fourier cosine transform
$\mathcal{F}_s[\cdot]$	Fourier sine transform
γ	Euler's constant, $\gamma = \int_0^\infty e^{-x} \operatorname{Log} x \, dx$
$\Gamma(\nu)$	Gamma function
$H(x)$	Heaviside function
$H_\nu^{(1)}(x)$	Hankel function of the first kind and order ν
$H_\nu^{(2)}(x)$	Hankel function of the second kind and order ν
$\imath$	$\imath \equiv \sqrt{-1}$
$J_\nu(x)$	Bessel function of the first kind and order ν
$K_\nu(x)$	Modified Bessel function of the first kind and order ν
$\mathcal{L}[\cdot]$	Laplace transform

$\mathbb{N}$	set of natural numbers, (positive integers)
$N_\nu(x)$	Modified Bessel function of the second kind and order ν
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}^-$	set of negative real numbers
$o(z)$	terms smaller than z
$\mathcal{O}(z)$	terms no bigger than z
$⨍$	principal value of the integral
$\psi(\nu)$	digamma function, $\psi(\nu) = \frac{\mathrm{d}}{\mathrm{d}\nu} \log \Gamma(\nu)$
$\psi^{(n)}(\nu)$	polygamma function, $\psi^{(n)}(\nu) = \frac{d^n}{d\nu^n}\psi(\nu)$
$u^{(n)}(x)$	$\frac{\partial^n u}{\partial x^n}$
$u^{(n,m)}(x,y)$	$\frac{\partial^{n+m} u}{\partial x^n \partial y^m}$
$Y_\nu(x)$	Bessel function of the second kind and order ν, Neumann function
$\mathbb{Z}$	set of integers
$\mathbb{Z}^+$	set of positive integers

Appendix C

Formulas from Complex Variables

Analytic Functions. A function $f(z)$ is analytic in a domain if the derivative $f'(z)$ exists in that domain.

If $f(z) = u(x, y) + \imath v(x, y)$ is defined in some neighborhood of $z_0 = x_0 + \imath y_0$ and the partial derivatives of u and v are continuous and satisfy the **Cauchy-Riemann equations**

$$u_x = v_y, \qquad u_y = -v_x,$$

then $f'(z_0)$ exists.

Residues. If $f(z)$ has the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

then the residue of $f(z)$ at $z = z_0$ is

$$\operatorname{Res}(f(z), z_0) = a_{-1}.$$

Residue Theorem. Let C be a positively oriented, simple, closed contour. If $f(z)$ is analytic in and on C except for isolated singularities at $z_1, z_2, \ldots, z_N$ inside C then

$$\oint_C f(z)\,dz = \imath 2\pi \sum_{n=1}^{N} \operatorname{Res}(f(z), z_n).$$

If in addition $f(z)$ is analytic outside C in the finite complex plane then

$$\oint_C f(z)\,dz = \imath 2\pi \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

Residues of a pole of order n. If $f(z)$ has a pole of order n at $z = z_0$ then

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} \left(\frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z - z_0)^n f(z)\right]\right).$$

Jordan's Lemma.

$$\int_0^{\pi} \mathrm{e}^{-R \sin\theta}\, \mathrm{d}\theta < \frac{\pi}{R}.$$

Let a be a positive constant. If $f(z)$ vanishes as $|z| \to \infty$ then the integral

$$\int_C f(z)\, \mathrm{e}^{\imath a z}\, \mathrm{d}z$$

along the semi-circle of radius R in the upper half plane vanishes as $R \to \infty$.

Taylor Series. Let $f(z)$ be a function that is analytic and single valued in the disk $|z - z_0| < R$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

The series converges for $|z - z_0| < R$.

Laurent Series. Let $f(z)$ be a function that is analytic and single valued in the annulus $r < |z - z_0| < R$. In this annulus $f(z)$ has the convergent series,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

where

$$c_n = \frac{1}{\imath 2\pi} \oint \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

and the path of integration is any simple, closed, positive contour around z_0 and lying in the annulus. The path of integration is shown in Figure C.1.

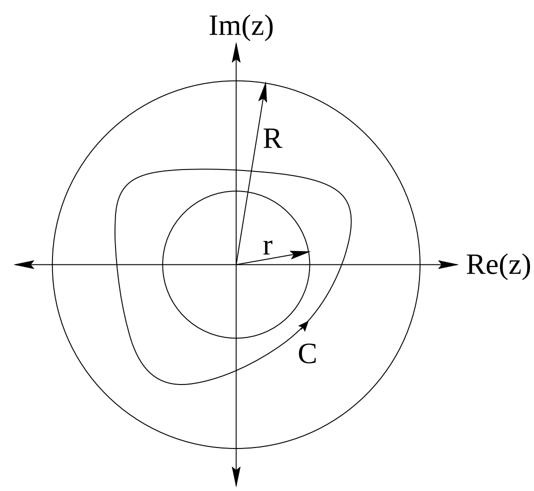

Figure C.1: The Path of Integration.

Appendix D

Table of Derivatives

Note: c denotes a constant and $'$ denotes differentiation.

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = \frac{\mathrm{d}f}{\mathrm{d}x}g + f\frac{\mathrm{d}g}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}f^c = cf^{c-1}f'$$

$$\frac{\mathrm{d}}{\mathrm{d}x}f(g) = f'(g)g'$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(g) = f''(g)(g')^2 + f'g''$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(fg) = \binom{n}{0}\frac{\mathrm{d}^n f}{\mathrm{d}x^n}g + \binom{n}{1}\frac{\mathrm{d}^{n-1} f}{\mathrm{d}x^{n-1}}\frac{\mathrm{d}g}{\mathrm{d}x} + \binom{n}{2}\frac{\mathrm{d}^{n-2} f}{\mathrm{d}x^{n-2}}\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + \cdots + \binom{n}{n}f\frac{\mathrm{d}^n g}{\mathrm{d}x^n}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{|x|}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}c^x = c^x \ln c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}f^g = gf^{g-1}\frac{\mathrm{d}f}{\mathrm{d}x} + f^g \ln f \frac{\mathrm{d}g}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \qquad -\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}}, \qquad 0 \le \arccos x \le \pi$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}, \qquad -\frac{\pi}{2} \le \arctan x \le \frac{\pi}{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh x = \operatorname{sech}^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\coth x = -\operatorname{csch}^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcsinh} x = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}}, \qquad x > 1,\ \operatorname{arccosh} x > 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arctanh} x = \frac{1}{1-x^2}, \qquad x^2 < 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_c^x f(\xi)\,d\xi = f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^c f(\xi)\,d\xi = -f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_g^h f(\xi,x)\,d\xi = \int_g^h \frac{\partial f(\xi,x)}{\partial x}\,d\xi + f(h,x)h' - f(g,x)g'$$

Appendix E

Table of Integrals

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\, dx = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\, dx$$

$$\int \frac{f'(x)}{f(x)}\, dx = \log f(x)$$

$$\int \frac{f'(x)}{2\sqrt{f(x)}}\, dx = \sqrt{f(x)}$$

$$\int x^{\alpha}\, dx = \frac{x^{\alpha+1}}{\alpha+1} \qquad \text{for } \alpha \neq = -1$$

$$\int \frac{1}{x}\, dx = \log x$$

$$\int \mathrm{e}^{ax}\, dx = \frac{\mathrm{e}^{ax}}{a}$$

$$\int a^{bx}\,dx = \frac{a^{bx}}{b\log a} \qquad \text{for } a > 0$$

$$\int \log x\,dx = x\log x - x$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}$$

$$\int \frac{1}{x^2-a^2}\,dx = \begin{cases} \frac{1}{2a}\log\frac{a-x}{a+x} & \text{for } x^2 < a^2 \\ \frac{1}{2a}\log\frac{x-a}{x+a} & \text{for } x^2 > a^2 \end{cases}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{|a|} = -\arccos\frac{x}{|a|} \qquad \text{for } x^2 < a^2$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \log(x+\sqrt{x^2\pm a^2})$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{|a|}\sec^{-1}\frac{x}{a}$$

$$\int \frac{1}{x\sqrt{a^2\pm x^2}}\,dx = -\frac{1}{a}\log\left(\frac{a+\sqrt{a^2\pm x^2}}{x}\right)$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)$$

$$\int \tan(ax)\,dx = -\frac{1}{a}\log\cos(ax)$$

$$\int \csc(ax)\,dx = \frac{1}{a}\log\tan\frac{ax}{2}$$

$$\int \sec(ax)\,dx = \frac{1}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$\int \cot(ax)\,dx = \frac{1}{a}\log\sin(ax)$$

$$\int \sinh(ax)\,dx = \frac{1}{a}\cosh(ax)$$

$$\int \cosh(ax)\,dx = \frac{1}{a}\sinh(ax)$$

$$\int \tanh(ax)\,dx = \frac{1}{a}\log\cosh(ax)$$

$$\int \operatorname{csch}(ax)\,dx = \frac{1}{a}\log\tanh\frac{ax}{2}$$

$$\int \operatorname{sech}(ax)\,dx = \frac{i}{a}\log\tanh\left(\frac{i\pi}{4} + \frac{ax}{2}\right)$$

$$\int \coth(ax)\,dx = \frac{1}{a}\log\sinh(ax)$$

$$\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$$

$$\int x^2 \sin ax \, dx = \frac{2x}{a^2} \sin ax - \frac{a^2x^2 - 2}{a^3} \cos ax$$

$$\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$$

$$\int x^2 \cos ax \, dx = \frac{2x \cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3} \sin ax$$

Appendix F

Definite Integrals

Integrals from $-\infty$ to ∞. Let $f(z)$ be analytic except for isolated singularities, none of which lie on the real axis. Let $a_1, \ldots, a_m$ be the singularities of $f(z)$ in the upper half plane; and C_R be the semi-circle from R to $-R$ in the upper half plane. If

$$\lim_{R \to \infty} \left(R \max_{z \in C_R} |f(z)| \right) = 0$$

then

$$\int_{-\infty}^{\infty} f(x)\,dx = \imath 2\pi \sum_{j=1}^{m} \operatorname{Res}\left(f(z), a_j\right).$$

Let $b_1, \ldots, b_n$ be the singularities of $f(z)$ in the lower half plane. Let C_R be the semi-circle from R to $-R$ in the lower half plane. If

$$\lim_{R \to \infty} \left(R \max_{z \in C_R} |f(z)| \right) = 0$$

then

$$\int_{-\infty}^{\infty} f(x)\,dx = -\imath 2\pi \sum_{j=1}^{n} \operatorname{Res}\left(f(z), b_j\right).$$

Integrals from 0 to ∞. Let $f(z)$ be analytic except for isolated singularities, none of which lie on the positive real axis, $[0, \infty)$. Let $z_1, \ldots, z_n$ be the singularities of $f(z)$. If $f(z) \ll z^{\alpha}$ as $z \to 0$ for some $\alpha > -1$ and $f(z) \ll z^{\beta}$ as $z \to \infty$ for some $\beta < -1$ then

$$\int_0^{\infty} f(x)\,dx = -\sum_{k=1}^{n} \operatorname{Res}\left(f(z)\log z, z_k\right).$$

$$\int_0^{\infty} f(x)\log\ dx = -\frac{1}{2}\sum_{k=1}^{n} \operatorname{Res}\left(f(z)\log^2 z, z_k\right) + \imath\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z)\log z, z_k\right)$$

Assume that a is not an integer. If $z^a f(z) \ll z^{\alpha}$ as $z \to 0$ for some $\alpha > -1$ and $z^a f(z) \ll z^{\beta}$ as $z \to \infty$ for some $\beta < -1$ then

$$\int_0^{\infty} x^a f(x)\,dx = \frac{\imath 2\pi}{1 - \mathrm{e}^{\imath 2\pi a}} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right).$$

$$\int_0^{\infty} x^a f(x)\log x\,dx = \frac{\imath 2\pi}{1 - \mathrm{e}^{\imath 2\pi a}} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z)\log z, z_k\right), + \frac{\pi^2 a}{\sin^2(\pi a)} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right)$$

Fourier Integrals. Let $f(z)$ be analytic except for isolated singularities, none of which lie on the real axis. Suppose that $f(z)$ vanishes as $|z| \to \infty$. If ω is a positive real number then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{\imath\omega x}\ dx = \imath 2\pi \sum_{k=1}^{n} \operatorname{Res}(f(z)\,\mathrm{e}^{\imath\omega z}, z_k),$$

where $z_1, \ldots, z_n$ are the singularities of $f(z)$ in the upper half plane. If ω is a negative real number then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{\imath\omega x}\ dx = -\imath 2\pi \sum_{k=1}^{n} \operatorname{Res}(f(z)\,\mathrm{e}^{\imath\omega z}, z_k),$$

where $z_1, \ldots, z_n$ are the singularities of $f(z)$ in the lower half plane.

Appendix G

Table of Sums

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \quad \text{for } |r| < 1$$

$$\sum_{n=1}^{N} r^n = \frac{r - r^{N+1}}{1-r}$$

$$\sum_{n=a}^{b} n = \frac{(a+b)(b+1-a)}{2}$$

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

$$\sum_{n=a}^{b} n^2 = \frac{b(b+1)(2b+1) - a(a-1)(2a-1)}{6}$$

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log(2)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = \frac{3\zeta(3)}{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5} = \zeta(5)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} = \frac{15\zeta(5)}{16}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} = \frac{31\pi^6}{30240}$$

Appendix H

Table of Taylor Series

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

$$(1-z)^{-2} = \sum_{n=0}^{\infty} (n+1) z^n \qquad |z| < 1$$

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \qquad |z| < 1$$

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad |z| < \infty$$

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \qquad |z| < 1$$

$$\log\left(\frac{1+z}{1-z}\right) = 2\sum_{n=1}^{\infty} \frac{z^{2n-1}}{2n-1} \qquad |z| < 1$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \qquad |z| < \infty$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad |z| < \infty$$

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \cdots \qquad |z| < \tfrac{\pi}{2}$$

$$\cos^{-1} z = \frac{\pi}{2} - \left(z + \frac{z^3}{2\cdot 3} + \frac{1\cdot 3z^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5z^7}{2\cdot 4\cdot 6\cdot 7} + \cdots\right) \qquad |z| < 1$$

$$\sin^{-1} z = z + \frac{z^3}{2\cdot 3} + \frac{1\cdot 3z^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5z^7}{2\cdot 4\cdot 6\cdot 7} + \cdots \qquad |z| < 1$$

$$\tan^{-1} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n-1}}{2n-1} \qquad |z| < 1$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad |z| < \infty$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad |z| < \infty$$

$$\tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} - \frac{17z^7}{315} + \cdots \qquad |z| < \tfrac{\pi}{2}$$

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{\nu+2n} \qquad |z| < \infty$$

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{\nu+2n} \qquad |z| < \infty$$

Appendix I

Continuous Transforms

I.1 Properties of Laplace Transforms

Let $f(t)$ be piecewise continuous and of exponential order α. Unless otherwise noted, the transform is defined for $s > 0$. To reduce clutter, it is understood that the Heaviside function $H(t)$ multiplies the original function in the following two tables.

$\mathbf{f(t)}$	$\int_0^\infty \mathrm{e}^{-\mathbf{st}}\, \mathbf{f(t)}\, \mathbf{dt}$
$\frac{1}{\imath 2\pi} \int_{c-\imath\infty}^{c+\imath\infty} \mathrm{e}^{ts}\, \hat{f}(s)\, ds$	$\hat{f}(s)$
$af(t) + bg(t)$	$a\hat{f}(s) + b\hat{g}(s)$
$\frac{\mathrm{d}}{\mathrm{d}t} f(t)$	$s\hat{f}(s) - f(0)$

$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)$	$s^2 \hat{f}(s) - s f(0) - f'(0)$	
$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)$	$s^n \hat{f}(s) - s^{n-1} f(0)$ $-s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$	
$\int_0^t f(\tau)\, d\tau$	$\frac{\hat{f}(s)}{s}$	
$\int_0^t \int_0^\tau f(s)\, ds\, d\tau$	$\frac{\hat{f}(s)}{s^2}$	
$\mathrm{e}^{ct} f(t)$	$\hat{f}(s-c)$	$s > c + \alpha$
$\frac{1}{c} f\left(\frac{t}{c}\right), \quad c > 0$	$\hat{f}(cs)$	
$\frac{1}{c} \mathrm{e}^{(b/c)t} f\left(\frac{t}{c}\right), \quad c > 0$	$\hat{f}(cs - b)$	
$f(t-c) H(t-c), \quad c > 0$	$\mathrm{e}^{-cs} \hat{f}(s)$	
$t f(t)$	$-\frac{\mathrm{d}}{\mathrm{d}s} \hat{f}(s)$	
$t^n f(t)$	$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} \hat{f}(s)$	
$\frac{f(t)}{t}, \quad \int_0^1 \frac{f(t)}{t}\, \mathrm{d}t$ exists	$\int_s^\infty \hat{f}(t)\, \mathrm{d}t$	

$$\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau, \quad f,g \in C^0 \quad \hat{f}(s)\hat{g}(s)$$

$$f(t), \quad f(t+T) = f(t) \qquad \frac{\int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t}{1-\mathrm{e}^{-sT}}$$

$$f(t), \quad f(t+T) = -f(t) \qquad \frac{\int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t}{1+\mathrm{e}^{-sT}}$$

I.2 Table of Laplace Transforms

$\mathbf{f(t)}$	$\int_{\mathbf{0}}^{\infty} \mathrm{e}^{-\mathbf{st}}\, \mathbf{f(t)}\, \mathbf{dt}$
$\frac{1}{\imath 2\pi} \int_{c-\imath\infty}^{c+\imath\infty} \mathrm{e}^{ts}\, \hat{f}(s)\, ds$	$\hat{f}(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n, for $n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
$t^{1/2}$	$\frac{\sqrt{\pi}}{2} s^{-3/2}$
$t^{-1/2}$	$\sqrt{\pi} s^{-1/2}$
$t^{n-1/2}, \quad n \in \mathbb{Z}^+$	$\frac{(1)(3)(5)\cdots(2n-1)\sqrt{\pi}}{2^n} s^{-n-1/2}$
$t^{\nu}, \quad \Re(\nu) > -1$	$\frac{\Gamma(\nu+1)}{s^{\nu+1}}$
$\operatorname{Log} t$	$\frac{-\gamma - \operatorname{Log} s}{s}$

$t^{\nu} \operatorname{Log} t, \quad \Re(\nu) > -1$	$\frac{\Gamma(\nu+1)}{s^{n+1}}(\psi(\nu+1) - \operatorname{Log} s)$	
$\delta(t)$	1	$s > 0$
$\delta^{(n)}(t), \quad n \in \mathbb{Z}^{0+}$	s^n	$s > 0$
e^{ct}	$\frac{1}{s-c}$	$s > c$
$t\,\mathrm{e}^{ct}$	$\frac{1}{(s-c)^2}$	$s > c$
$\frac{t^{n-1}\,\mathrm{e}^{ct}}{(n-1)!}, n \in \mathbb{Z}^+$	$\frac{1}{(s-c)^n}$	$s > c$
$\sin(ct)$	$\frac{c}{s^2+c^2}$	
$\cos(ct)$	$\frac{s}{s^2+c^2}$	
$\sinh(ct)$	$\frac{c}{s^2-c^2}$	$s > \|c\|$
$\cosh(ct)$	$\frac{s}{s^2-c^2}$	$s > \|c\|$
$t\sin(ct)$	$\frac{2cs}{(s^2+c^2)^2}$	
$t\cos(ct)$	$\frac{s^2-c^2}{(s^2+c^2)^2}$	

$t^n\,\mathrm{e}^{ct}, \quad n \in \mathbb{Z}^+$	$\dfrac{n!}{(s-c)^{n+1}}$	
$\mathrm{e}^{dt}\sin(ct)$	$\dfrac{c}{(s-d)^2+c^2}$	$s > d$
$\mathrm{e}^{dt}\cos(ct)$	$\dfrac{s-d}{(s-d)^2+c^2}$	$s > d$
$\delta(t-c)$	$\begin{cases} 0 & \text{for } c < 0 \\ \mathrm{e}^{-sc} & \text{for } c > 0 \end{cases}$	
$H(t-c) = \begin{cases} 0 & \text{for } t < c \\ 1 & \text{for } t > c \end{cases}$	$\dfrac{1}{s}\,\mathrm{e}^{-cs}$	
$J_\nu(ct)$	$\dfrac{c^n}{\sqrt{s^2+c^2}\left(s+\sqrt{s^2+c^2}\right)^\nu}$	$\nu > -1$
$I_\nu(ct)$	$\dfrac{c^n}{\sqrt{s^2-c^2}\left(s-\sqrt{s^2+c^2}\right)^\nu}$	$\Re(s) > c,\ \nu > -1$

I.3 Table of Fourier Transforms

$\mathbf{f(x)}$	$\frac{\mathbf{1}}{\mathbf{2\pi}}\int_{-\infty}^{\infty}\mathbf{f(x)}\,\mathrm{e}^{-\imath\omega\mathbf{x}}\,\mathrm{d}\mathbf{x}$
$\int_{-\infty}^{\infty}\mathbf{F}(\boldsymbol{\omega})\,\mathrm{e}^{\imath\omega\mathbf{x}}\,\mathrm{d}\boldsymbol{\omega}$	$\mathbf{F}(\omega)$
$af(x)+bg(x)$	$aF(\omega)+bG(\omega)$
$f^{(n)}(x)$	$(\imath\omega)^n F(\omega)$
$x^n f(x)$	$\imath^n F^{(n)}(\omega)$
$f(x+c)$	$\mathrm{e}^{\imath\omega c}\,F(\omega)$
$\mathrm{e}^{-\imath cx}\,f(x)$	$F(\omega+c)$
$f(cx)$	$\lvert c\rvert^{-1}F(\omega/c)$
$f(x)g(x)$	$F*G(\omega)=\int_{-\infty}^{\infty}F(\eta)G(\omega-\eta)\,\mathrm{d}\eta$
$\frac{1}{2\pi}f*g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)g(x-\xi)\,\mathrm{d}\xi$	$F(\omega)G(\omega)$
$\mathrm{e}^{-cx^2}, \quad c>0$	$\frac{1}{\sqrt{4\pi c}}\,\mathrm{e}^{-\omega^2/4c}$

$\mathrm{e}^{-c\|x\|}, \quad c > 0$	$\frac{c/\pi}{\omega^2 + c^2}$
$\frac{2c}{x^2 + c^2}, \quad c > 0$	$\mathrm{e}^{-c\|\omega\|}$
$\frac{1}{x - \imath\alpha}, \quad \alpha > 0$	$\begin{cases} 0 & \text{for } \omega > 0 \\ \imath\, \mathrm{e}^{\alpha\omega} & \text{for } \omega < 0 \end{cases}$
$\frac{1}{x - \imath\alpha}, \quad \alpha < 0$	$\begin{cases} \imath\, \mathrm{e}^{\alpha\omega} & \text{for } \omega > 0 \\ 0 & \text{for } \omega < 0 \end{cases}$
$\frac{1}{x}$	$-\frac{\imath}{2} \operatorname{sign}(\omega)$
$H(x - c) = \begin{cases} 0 & \text{for } x < c \\ 1 & \text{for } x > c \end{cases}$	$\frac{1}{\imath 2\pi\omega}\, \mathrm{e}^{-\imath c\omega}$
$\mathrm{e}^{-cx}\, H(x), \quad \Re(c) > 0$	$\frac{1}{2\pi(c + \imath\omega)}$
$\mathrm{e}^{cx}\, H(-x), \quad \Re(c) > 0$	$\frac{1}{2\pi(c - \imath\omega)}$
1	$\delta(\omega)$
$\delta(x - \xi)$	$\frac{1}{2\pi}\, \mathrm{e}^{-\imath\omega\xi}$
$\pi(\delta(x + \xi) + \delta(x - \xi))$	$\cos(\omega\xi)$

$-\imath\pi(\delta(x+\xi)-\delta(x-\xi))$	$\sin(\omega\xi)$
$H(c-\lvert x\rvert)=\begin{cases}1 & \text{for } \lvert x\rvert < c\\ 0 & \text{for } \lvert x\rvert > c\end{cases},\ c>0$	$\dfrac{\sin(c\omega)}{\pi\omega}$

I.4 Table of Fourier Transforms in n Dimensions

$\mathbf{f(x)}$	$\frac{\mathbf{1}}{(\mathbf{2}\boldsymbol{\pi})^{\mathbf{n}}}\int_{\mathbb{R}^{\mathbf{n}}} \mathbf{f(x)}\, \mathrm{e}^{-\boldsymbol{\imath\omega}\mathbf{x}}\, \mathrm{d}\mathbf{x}$
$\int_{\mathbb{R}^{\mathbf{n}}} \mathbf{F}(\boldsymbol{\omega})\, \mathrm{e}^{\boldsymbol{\imath\omega}\mathbf{x}}\, \mathrm{d}\boldsymbol{\omega}$	$\mathbf{F}(\omega)$
$af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
$\left(\frac{\pi}{c}\right)^{n/2} \mathrm{e}^{-nx^2/4c}$	$\mathrm{e}^{-c\omega^2}$

I.5 Table of Fourier Cosine Transforms

$\mathbf{f(x)}$	$\boldsymbol{\frac{1}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx}$
$\boldsymbol{2\int_0^\infty C(\omega)\cos(\omega x)\,d\omega}$	$\mathbf{C}(\boldsymbol{\omega})$
$f'(x)$	$\omega S(\omega) - \frac{1}{\pi}f(0)$
$f''(x)$	$-\omega^2 C(\omega) - \frac{1}{\pi}f'(0)$
$xf(x)$	$\frac{\partial}{\partial\omega}\mathcal{F}_s[f(x)]$
$f(cx), \quad c > 0$	$\frac{1}{c}C\left(\frac{\omega}{c}\right)$
$\frac{2c}{x^2+c^2}$	$\mathrm{e}^{-c\omega}$
e^{-cx}	$\frac{c/\pi}{\omega^2+c^2}$
e^{-cx^2}	$\frac{1}{\sqrt{4\pi c}}\,\mathrm{e}^{-\omega^2/(4c)}$
$\sqrt{\frac{\pi}{c}}\,\mathrm{e}^{-x^2/(4c)}$	$\mathrm{e}^{-c\omega^2}$

I.6 Table of Fourier Sine Transforms

$\mathbf{f(x)}$	$\frac{1}{\pi}\int_0^\infty \mathbf{f(x)} \sin(\omega x)\, dx$
$2\int_0^\infty \mathbf{S}(\omega) \sin(\omega x)\, d\omega$	$\mathbf{S}(\omega)$
$f'(x)$	$-\omega C(\omega)$
$f''(x)$	$-\omega^2 S(\omega) + \frac{1}{\pi}\omega f(0)$
$x f(x)$	$-\frac{\partial}{\partial\omega}\mathcal{F}_c[f(x)]$
$f(cx), \quad c > 0$	$\frac{1}{c} S\left(\frac{\omega}{c}\right)$
$\frac{2x}{x^2 + c^2}$	$e^{-c\omega}$
e^{-cx}	$\frac{\omega/\pi}{\omega^2 + c^2}$
$2\arctan\left(\frac{x}{c}\right)$	$\frac{1}{\omega} e^{-c\omega}$
$\frac{1}{x} e^{-cx}$	$\frac{1}{\pi}\arctan\left(\frac{\omega}{c}\right)$

1	$\frac{1}{\pi\omega}$
$\frac{2}{x}$	1
$x\,\mathrm{e}^{-cx^2}$	$\frac{\omega}{4c^{3/2}\sqrt{\pi}}\,\mathrm{e}^{-\omega^2/(4c)}$
$\frac{\sqrt{\pi}x}{2c^{3/2}}\,\mathrm{e}^{-x^2/(4c)}$	$\omega\,\mathrm{e}^{-c\omega^2}$

Appendix J

Table of Wronskians

$W\left[x-a, x-b\right]$	$b-a$
$W\left[\mathrm{e}^{ax}, \mathrm{e}^{bx}\right]$	$(b-a)\,\mathrm{e}^{(a+b)x}$
$W\left[\cos(ax), \sin(ax)\right]$	a
$W\left[\cosh(ax), \sinh(ax)\right]$	a
$W\left[\mathrm{e}^{ax}\cos(bx), \mathrm{e}^{ax}\sin(bx)\right]$	$b\,\mathrm{e}^{2ax}$
$W\left[\mathrm{e}^{ax}\cosh(bx), \mathrm{e}^{ax}\sinh(bx)\right]$	$b\,\mathrm{e}^{2ax}$
$W\left[\sin(c(x-a)), \sin(c(x-b))\right]$	$c\sin(c(b-a))$
$W\left[\cos(c(x-a)), \cos(c(x-b))\right]$	$c\sin(c(b-a))$
$W\left[\sin(c(x-a)), \cos(c(x-b))\right]$	$-c\cos(c(b-a))$

$W\left[\sinh(c(x-a)), \sinh(c(x-b))\right]$	$c\sinh(c(b-a))$
$W\left[\cosh(c(x-a)), \cosh(c(x-b))\right]$	$c\cosh(c(b-a))$
$W\left[\sinh(c(x-a)), \cosh(c(x-b))\right]$	$-c\cosh(c(b-a))$
$W\left[\mathrm{e}^{dx}\sin(c(x-a)), \mathrm{e}^{dx}\sin(c(x-b))\right]$	$c\,\mathrm{e}^{2dx}\sin(c(b-a))$
$W\left[\mathrm{e}^{dx}\cos(c(x-a)), \mathrm{e}^{dx}\cos(c(x-b))\right]$	$c\,\mathrm{e}^{2dx}\sin(c(b-a))$
$W\left[\mathrm{e}^{dx}\sin(c(x-a)), \mathrm{e}^{dx}\cos(c(x-b))\right]$	$-c\,\mathrm{e}^{2dx}\cos(c(b-a))$
$W\left[\mathrm{e}^{dx}\sinh(c(x-a)), \mathrm{e}^{dx}\sinh(c(x-b))\right]$	$c\,\mathrm{e}^{2dx}\sinh(c(b-a))$
$W\left[\mathrm{e}^{dx}\cosh(c(x-a)), \mathrm{e}^{dx}\cosh(c(x-b))\right]$	$-c\,\mathrm{e}^{2dx}\sinh(c(b-a))$
$W\left[\mathrm{e}^{dx}\sinh(c(x-a)), \mathrm{e}^{dx}\cosh(c(x-b))\right]$	$-c\,\mathrm{e}^{2dx}\cosh(c(b-a))$
$W\left[(x-a)\,\mathrm{e}^{cx}, (x-b)\,\mathrm{e}^{cx}\right]$	$(b-a)\,\mathrm{e}^{2cx}$

Appendix K

Sturm-Liouville Eigenvalue Problems

- $y'' + \lambda^2 y = 0$, $y(a) = y(b) = 0$

$$\lambda_n = \frac{n\pi}{b-a}, \quad y_n = \sin\left(\frac{n\pi(x-a)}{b-a}\right), \quad n \in \mathbb{N}$$

$$\langle y_n, y_n \rangle = \frac{b-a}{2}$$

- $y'' + \lambda^2 y = 0$, $y(a) = y'(b) = 0$

$$\lambda_n = \frac{(2n-1)\pi}{2(b-a)}, \quad y_n = \sin\left(\frac{(2n-1)\pi(x-a)}{2(b-a)}\right), \quad n \in \mathbb{N}$$

$$\langle y_n, y_n \rangle = \frac{b-a}{2}$$

- $y'' + \lambda^2 y = 0$, $y'(a) = y(b) = 0$

$$\lambda_n = \frac{(2n-1)\pi}{2(b-a)}, \quad y_n = \cos\left(\frac{(2n-1)\pi(x-a)}{2(b-a)}\right), \quad n \in \mathbb{N}$$

$$\langle y_n, y_n \rangle = \frac{b-a}{2}$$

- $y'' + \lambda^2 y = 0,\ y'(a) = y'(b) = 0$

$$\lambda_n = \frac{n\pi}{b-a}, \quad y_n = \cos\left(\frac{n\pi(x-a)}{b-a}\right), \quad n = 0, 1, 2, \ldots$$

$$\langle y_0, y_0 \rangle = b - a, \quad \langle y_n, y_n \rangle = \frac{b-a}{2} \text{ for } n \in \mathbb{N}$$

Appendix L

Green Functions for Ordinary Differential Equations

- $G' + p(x)G = \delta(x - \xi)$, $G(\xi^- : \xi) = 0$

$$G(x|\xi) = \exp\left(-\int_\xi^x p(t)\,\mathrm{d}t\right) H(x - \xi)$$

- $y'' = 0$, $y(a) = y(b) = 0$

$$G(x|\xi) = \frac{(x_< - a)(x_> - b)}{b - a}$$

- $y'' = 0$, $y(a) = y'(b) = 0$

$$G(x|\xi) = a - x_<$$

- $y'' = 0$, $y'(a) = y(b) = 0$

$$G(x|\xi) = x_> - b$$

- $y'' - c^2 y = 0$, $y(a) = y(b) = 0$

$$G(x|\xi) = \frac{\sinh(c(x_< - a)) \sinh(c(x_> - b))}{c \sinh(c(b - a))}$$

- $y'' - c^2 y = 0$, $y(a) = y'(b) = 0$

$$G(x|\xi) = -\frac{\sinh(c(x_< - a)) \cosh(c(x_> - b))}{c \cosh(c(b - a))}$$

- $y'' - c^2 y = 0$, $y'(a) = y(b) = 0$

$$G(x|\xi) = \frac{\cosh(c(x_< - a)) \sinh(c(x_> - b))}{c \cosh(c(b - a))}$$

- $y'' + c^2 y = 0$, $y(a) = y(b) = 0$, $c \neq \frac{npi}{b-a}$, $n \in \mathbb{N}$

$$G(x|\xi) = \frac{\sin(c(x_< - a)) \sin(c(x_> - b))}{c \sin(c(b - a))}$$

- $y'' + c^2 y = 0$, $y(a) = y'(b) = 0$, $c \neq \frac{(2n-1)pi}{2(b-a)}$, $n \in \mathbb{N}$

$$G(x|\xi) = -\frac{\sin(c(x_< - a)) \cos(c(x_> - b))}{c \cos(c(b - a))}$$

- $y'' + c^2 y = 0$, $y'(a) = y(b) = 0$, $c \neq \frac{(2n-1)pi}{2(b-a)}$, $n \in \mathbb{N}$

$$G(x|\xi) = \frac{\cos(c(x_< - a)) \sin(c(x_> - b))}{c \cos(c(b - a))}$$

- $y'' + 2cy' + dy = 0,\ y(a) = y(b) = 0,\ c^2 > d$

$$G(x|\xi) = \frac{\mathrm{e}^{-cx_<} \sinh(\sqrt{c^2 - d}(x_< - a))\, \mathrm{e}^{-cx_<} \sinh(\sqrt{c^2 - d}(x_> - b))}{\sqrt{c^2 - d}\, \mathrm{e}^{-2c\xi} \sinh(\sqrt{c^2 - d}(b - a))}$$

- $y'' + 2cy' + dy = 0,\ y(a) = y(b) = 0,\ c^2 < d,\ \sqrt{d - c^2} \neq \frac{n\pi}{b-a},\ n \in \mathbb{N}$

$$G(x|\xi) = \frac{\mathrm{e}^{-cx_<} \sin(\sqrt{d - c^2}(x_< - a))\, \mathrm{e}^{-cx_<} \sin(\sqrt{d - c^2}(x_> - b))}{\sqrt{d - c^2}\, \mathrm{e}^{-2c\xi} \sin(\sqrt{d - c^2}(b - a))}$$

- $y'' + 2cy' + dy = 0,\ y(a) = y(b) = 0,\ c^2 = d$

$$G(x|\xi) = \frac{(x_< - a)\, \mathrm{e}^{-cx_<} (x_> - b)\, \mathrm{e}^{-cx_<}}{(b - a)\, \mathrm{e}^{-2c\xi}}$$

Appendix M

Trigonometric Identities

M.1 Circular Functions

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1, \qquad 1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x$$

Angle Sum and Difference Identities

$$\begin{aligned}
\sin(x+y) &= \sin x \cos y + \cos x \sin y \\
\sin(x-y) &= \sin x \cos y - \cos x \sin y \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
\cos(x-y) &= \cos x \cos y + \sin x \sin y
\end{aligned}$$

Function Sum and Difference Identities

$$\sin x + \sin y = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$$
$$\sin x - \sin y = 2 \cos \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$$
$$\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$$
$$\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$$

Double Angle Identities

$$\sin 2x = 2 \sin x \cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Half Angle Identities

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}, \qquad \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$$

Function Product Identities

$$\sin x \sin y = \frac{1}{2} \cos(x - y) - \frac{1}{2} \cos(x + y)$$
$$\cos x \cos y = \frac{1}{2} \cos(x - y) + \frac{1}{2} \cos(x + y)$$
$$\sin x \cos y = \frac{1}{2} \sin(x + y) + \frac{1}{2} \sin(x - y)$$
$$\cos x \sin y = \frac{1}{2} \sin(x + y) - \frac{1}{2} \sin(x - y)$$

Exponential Identities

$$\mathrm{e}^{\imath x} = \cos x + \imath \sin x, \qquad \sin x = \frac{\mathrm{e}^{\imath x} - \mathrm{e}^{-\imath x}}{\imath 2}, \qquad \cos x = \frac{\mathrm{e}^{\imath x} + \mathrm{e}^{-\imath x}}{2}$$

M.2 Hyperbolic Functions

Exponential Identities

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Reciprocal Identities

$$\operatorname{csch} x = \frac{1}{\sinh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}$$

Pythagorean Identities

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1$$

Relation to Circular Functions

$$\begin{aligned} \sinh(\imath x) &= \imath \sin x & \sinh x &= -\imath \sin(\imath x) \\ \cosh(\imath x) &= \cos x & \cosh x &= \cos(\imath x) \\ \tanh(\imath x) &= \imath \tan x & \tanh x &= -\imath \tan(\imath x) \end{aligned}$$

Angle Sum and Difference Identities

$$\begin{aligned} \sinh(x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y \\ \cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y \\ \tanh(x \pm y) &= \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} = \frac{\sinh 2x \pm \sinh 2y}{\cosh 2x \pm \cosh 2y} \\ \coth(x \pm y) &= \frac{1 \pm \coth x \coth y}{\coth x \pm \coth y} = \frac{\sinh 2x \mp \sinh 2y}{\cosh 2x - \cosh 2y} \end{aligned}$$

Function Sum and Difference Identities

$$\sinh x \pm \sinh y = 2 \sinh \frac{1}{2}(x \pm y) \cosh \frac{1}{2}(x \mp y)$$
$$\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$$
$$\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$$
$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$$
$$\coth x \pm \coth y = \frac{\sinh(x \pm y)}{\sinh x \sinh y}$$

Double Angle Identities

$$\sinh 2x = 2 \sinh x \cosh x, \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

Half Angle Identities

$$\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2}, \qquad \cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$$

Function Product Identities

$$\sinh x \sinh y = \frac{1}{2} \cosh(x + y) - \frac{1}{2} \cosh(x - y)$$
$$\cosh x \cosh y = \frac{1}{2} \cosh(x + y) + \frac{1}{2} \cosh(x - y)$$
$$\sinh x \cosh y = \frac{1}{2} \sinh(x + y) + \frac{1}{2} \sinh(x - y)$$

See Figure M.1 for plots of the hyperbolic circular functions.

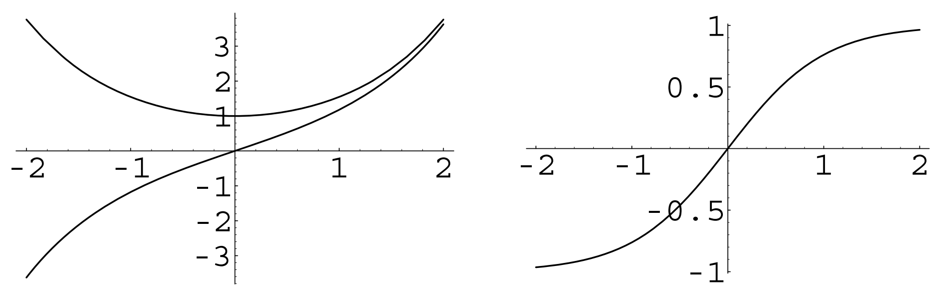

Figure M.1: $\cosh x$, $\sinh x$ and then $\tanh x$

Appendix N

Bessel Functions

N.1 Definite Integrals

Let $\nu > -1$.

$$\int_0^1 r J_\nu(j_{\nu,m} r) J_\nu(j_{\nu,n} r)\,\mathrm{d}r = \frac{1}{2}\left(J'_\nu(j_{\nu,n})\right)^2 \delta_{mn}$$

$$\int_0^1 r J_\nu(j'_{\nu,m} r) J_\nu(j'_{\nu,n} r)\,\mathrm{d}r = \frac{j'^2_{\nu,n} - \nu^2}{2 j'^2_{\nu,n}} \left(J_\nu(j'_{\nu,n})\right)^2 \delta_{mn}$$

$$\int_0^1 r J_\nu(\alpha_m r) J_\nu(\alpha_n r)\,\mathrm{d}r = \frac{1}{2\alpha_n^2}\left(\frac{a^2}{b^2} + \alpha_n^2 - \nu^2\right) \left(J_\nu(\alpha_n)\right)^2 \delta_{mn}$$

Here α_n is the n^{th} positive root of $aJ_\nu(r) + brJ'_\nu(r)$, where $a, b \in \mathbb{R}$.

Appendix O

Formulas from Linear Algebra

Kramer's Rule. Consider the matrix equation

$$A\vec{x} = \vec{b}.$$

This equation has a unique solution if and only if $\det(A) \neq 0$. If the determinant vanishes then there are either no solutions or an infinite number of solutions. If the determinant is nonzero, the solution for each x_j can be written

$$x_j = \frac{\det A_j}{\det A}$$

where A_j is the matrix formed by replacing the j^{th} column of A with b.

Example O.0.1 *The matrix equation*

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix},$$

has the solution

$$x_1 = \frac{\begin{vmatrix} 5 & 2 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{8}{-2} = -4, \qquad x_2 = \frac{\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{-9}{-2} = \frac{9}{2}.$$

Appendix P

Vector Analysis

Rectangular Coordinates

$$f = f(x, y, z), \qquad \vec{g} = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k}$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\nabla \cdot \vec{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$$

$$\nabla \times \vec{g} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_x & g_y & g_z \end{vmatrix}$$

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi$$

$$f = f(r,\theta,\phi), \qquad \vec{g} = g_r\mathbf{r} + g_\theta\boldsymbol{\theta} + g_\phi\boldsymbol{\phi}$$

Divergence Theorem.

$$\iint \nabla\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}y = \oint \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s$$

Stoke's Theorem.

$$\iint (\nabla\times\mathbf{u})\cdot\,\mathrm{d}\mathbf{s} = \oint \mathbf{u}\cdot\,\mathrm{d}\mathbf{r}$$

Appendix Q

Partial Fractions

A proper rational function

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x-a)^n r(x)}$$

Can be written in the form

$$\frac{p(x)}{(x-\alpha)^n r(x)} = \left(\frac{a_0}{(x-\alpha)^n} + \frac{a_1}{(x-\alpha)^{n-1}} + \cdots + \frac{a_{n-1}}{x-\alpha} \right) + (\cdots)$$

where the a_k's are constants and the last ellipses represents the partial fractions expansion of the roots of $r(x)$. The coefficients are

$$a_k = \frac{1}{k!} \frac{d^k}{dx^k} \left(\frac{p(x)}{r(x)} \right) \bigg|_{x=\alpha}.$$

Example Q.0.2 *Consider the partial fraction expansion of*

$$\frac{1+x+x^2}{(x-1)^3}.$$

The expansion has the form

$$\frac{a_0}{(x-1)^3} + \frac{a_1}{(x-1)^2} + \frac{a_2}{x-1}.$$

The coefficients are

$$
\begin{aligned}
a_0 &= \frac{1}{0!}(1+x+x^2)|_{x=1} = 3,\\
a_1 &= \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}x}(1+x+x^2)|_{x=1} = (1+2x)|_{x=1} = 3,\\
a_2 &= \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1+x+x^2)|_{x=1} = \frac{1}{2}(2)|_{x=1} = 1.
\end{aligned}
$$

Thus we have

$$\frac{1+x+x^2}{(x-1)^3} = \frac{3}{(x-1)^3} + \frac{3}{(x-1)^2} + \frac{1}{x-1}.$$

Example Q.0.3 *Consider the partial fraction expansion of*

$$\frac{1+x+x^2}{x^2(x-1)^2}.$$

The expansion has the form

$$\frac{a_0}{x^2} + \frac{a_1}{x} + \frac{b_0}{(x-1)^2} + \frac{b_1}{x-1}.$$

The coefficients are

$$
\begin{aligned}
a_0 &= \frac{1}{0!}\left(\frac{1+x+x^2}{(x-1)^2}\right)\bigg|_{x=0} = 1,\\
a_1 &= \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x+x^2}{(x-1)^2}\right)\bigg|_{x=0} = \left(\frac{1+2x}{(x-1)^2} - \frac{2(1+x+x^2)}{(x-1)^3}\right)\bigg|_{x=0} = 3,\\
b_0 &= \frac{1}{0!}\left(\frac{1+x+x^2}{x^2}\right)\bigg|_{x=1} = 3,\\
b_1 &= \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x+x^2}{x^2}\right)\bigg|_{x=1} = \left(\frac{1+2x}{x^2} - \frac{2(1+x+x^2)}{x^3}\right)\bigg|_{x=1} = -3,
\end{aligned}
$$

Thus we have

$$\frac{1+x+x^2}{x^2(x-1)^2} = \frac{1}{x^2} + \frac{3}{x} + \frac{3}{(x-1)^2} - \frac{3}{x-1}.$$

If the rational function has real coefficients and the denominator has complex roots, then you can reduce the work in finding the partial fraction expansion with the following trick: Let α and $\overline{\alpha}$ be complex conjugate pairs of roots of the denominator.

$$\begin{aligned}\frac{p(x)}{(x-\alpha)^n(x-\overline{\alpha})^n r(x)} = &\left(\frac{a_0}{(x-\alpha)^n} + \frac{a_1}{(x-\alpha)^{n-1}} + \cdots + \frac{a_{n-1}}{x-\alpha}\right) \\ &+ \left(\frac{\overline{a_0}}{(x-\overline{\alpha})^n} + \frac{\overline{a_1}}{(x-\overline{\alpha})^{n-1}} + \cdots + \frac{\overline{a_{n-1}}}{x-\overline{\alpha}}\right) + (\cdots)\end{aligned}$$

Thus we don't have to calculate the coefficients for the root at $\overline{\alpha}$. We just take the complex conjugate of the coefficients for α.

Example Q.0.4 *Consider the partial fraction expansion of*

$$\frac{1+x}{x^2+1}.$$

The expansion has the form

$$\frac{a_0}{x-i} + \frac{\overline{a_0}}{x+i}$$

The coefficients are

$$\begin{aligned} a_0 &= \frac{1}{0!}\left(\frac{1+x}{x+i}\right)\bigg|_{x=i} = \frac{1}{2}(1-i), \\ \overline{a_0} &= \overline{\frac{1}{2}(1-i)} = \frac{1}{2}(1+i) \end{aligned}$$

Thus we have

$$\frac{1+x}{x^2+1} = \frac{1-i}{2(x-i)} + \frac{1+i}{2(x+i)}.$$

Appendix R

Finite Math

Newton's Binomial Formula.

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{k}{n} a^{n-k} b^k \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2} a^{n-2} b^2 + \cdots + nab^{n-1} + b^n,\end{aligned}$$

The *binomial coefficients* are,

$$\binom{k}{n} = \frac{n!}{k!(n-k)!}.$$